Series of geometric inequalities.
https://www.linkedin.com/feed/update/urn:li:activity:6478227775303098368
In any triangle with usual notions, prove that
(**8**) $36r^2 \leq a^2+b^2+c^2$;
(**9**) $4r^2 \leq \dfrac{abc}{a+b+c}$;
(**10**) $\dfrac{1}{R^2} \leq \dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca}$;
(**11**) $\dfrac{\sqrt{3}}{R} \leq \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$;
(**12**) $16Rr-5r^2 \leq s^2$;
(**13**) $4r(5R-r) \leq ab+bc+ca$;
(**14**) $a(s-a)+b(s-b)+c(s-c) \leq 9Rr$;
(**15**) $\dfrac{1}{2Rr} \leq \dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \leq \dfrac{1}{4r^2}$.
Solution by Arkady Alt , **San Jose** ,**California**, **USA**.
Let $x := s-a, y := s-b, z := s-c, p := xy+yz+zx, q := xyz$. Then, assuming $s=1$ (due homogeneity), we obtain $x,y,z>0, x+y+z=1, a=1-x, b=1-y, c=1-z$ and $ab+bc+ca=1+p, a^2+b^2+c^2=2(1-p), abc=p-q, r=\sqrt{q}, R=\dfrac{p-q}{4\sqrt{q}}$.

In that notation can be useful the following inequalities:
(**1**) $3p \leq 1 (3p = 3(xy+yz+zx) \leq (x+y+z)^2 = 1)$;
(**2**) $3q \leq p^2$ ($3q = 3xyz(x+y+z) \leq (xy+yz+zx)^2 = p^2$);
(**3**) $9q \leq p$ (since $3q \leq p^2 \leq p \cdot \dfrac{1}{3}$)
(**4**) $9q \geq 4p-1$(it is Schure's Inequality $\sum x(x-y)(x-z) \geq 0$ in p,q-notation and normalized by $x+y+z=1$)
(**5**) $q \leq \dfrac{p^2}{4-3p}$(it is inequality $\sum x^3(y-z)^2 \geq 0$ in p,q-notation and normalized by $x+y+z=1$).
We did that preparation because in the such parametrization of a triangle geometric inequality can be equivalently transformed to well known algebraic inequality.
For example inequality (**8**):
$36r^2 \leq a^2+b^2+c^2 \Leftrightarrow 36q \leq 2(1-p) \Leftrightarrow 18q \leq 1-p$ and
$1-p-18q \geq 1-p-18 \cdot \dfrac{p^2}{3} = (2p+1)(1-3p) \geq 0.$
Inequalities (**9**) and (**10**) are equivalent to Euler's Inequality $2r \leq R$.
Indeed, $4r^2 \leq \dfrac{abc}{a+b+c} \Leftrightarrow 4r^2 \leq \dfrac{4Rrs}{2s} \Leftrightarrow 2r \leq R$ and
$\dfrac{1}{R^2} \leq \dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca} \Leftrightarrow \dfrac{1}{R^2} \leq \dfrac{a+b+c}{abc} \Leftrightarrow \dfrac{1}{R^2} \leq \dfrac{2s}{4Rrs} \Leftrightarrow 2r \leq R.$
And $2r \leq R \Leftrightarrow 2\sqrt{q} \leq \dfrac{p-q}{4\sqrt{q}} \Leftrightarrow 9q \leq p.$
(**11**) $\dfrac{\sqrt{3}}{R} \leq \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \Leftrightarrow \dfrac{\sqrt{3}}{R} \leq \dfrac{ab+bc+ca}{abc} \Leftrightarrow \dfrac{\sqrt{3}}{R} \leq \dfrac{ab+bc+ca}{4Rrs} \Leftrightarrow$
$\sqrt{3q} \leq p \Leftrightarrow 3q \leq p^2.$
(**12**) (Gerretsen's Inequality) $16Rr-5r^2 \leq s^2 \Leftrightarrow 16 \cdot \dfrac{p-q}{4} - 5q \leq 1 \Leftrightarrow 9q \geq 4p-1.$

(13) $4r(5R-r) \le ab+bc+ca \Leftrightarrow 20Rr-4r^2 \le ab+bc+ca \Leftrightarrow$
$5(p-q)-9q \le p \Leftrightarrow 9q \ge 4p-1$
(or $4r(5R-r) \le ab+bc+ca = s^2+4Rr+r^2 \Leftrightarrow 16Rr-5r^2 \le s^2$).

(14) $\sum a(s-a) \le 9Rr \Leftrightarrow \sum(1-x)x \le \dfrac{9(p-q)}{4} \Leftrightarrow 2p \le \dfrac{9(p-q)}{4} \Leftrightarrow$
$8p \le 9(p-q) \Leftrightarrow 9q \le p.$

(15)

a) $\dfrac{1}{2Rr} \le \dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \Leftrightarrow \dfrac{1}{2Rr} \le \dfrac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} \Leftrightarrow$
$\dfrac{1}{2Rr} \le \dfrac{a^2b^2+b^2c^2+c^2a^2}{16R^2r^2s^2} \Leftrightarrow 8Rrs^2 \le (ab+bc+ca)^2-2abc(a+b+c) \Leftrightarrow$
$2(p-q) \le (1+p)^2-4(p-q) \Leftrightarrow 6(p-q) \le (1+p)^2$ and
$(1+p)^2-6\left(p-\dfrac{4p-1}{9}\right) = \dfrac{(1-3p)(1-p)}{3} \ge 0;$

b) $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \le \dfrac{1}{4r^2} \Leftrightarrow \dfrac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} \le \dfrac{1}{4r^2} \Leftrightarrow$
$\dfrac{a^2b^2+b^2c^2+c^2a^2}{16R^2r^2s^2} \le \dfrac{1}{4r^2} \Leftrightarrow (ab+bc+ca)^2-2abc(a+b+c) \le 4R^2s^2 \Leftrightarrow$
$(1+p)^2-4(p-q) \le \dfrac{(p-q)^2}{4q}$. Since $\dfrac{(p-q)^2}{4q}+4(p-q)$ decrease by $q \in \left[0, \dfrac{p^2}{4-3p}\right]$
then $\dfrac{(p-q)^2}{4q}+4(p-q)-(1+p)^2 \ge \dfrac{(p-p^2/(4-3p))^2}{4 \cdot p^2/(4-3p)}+4(p-p^2/(4-3p))-(1+p)^2 =$
$\dfrac{p(1-3p)(3-p)}{4-3p} \ge 0.$

(Note that upper bound $p^2/3$ for q isn't good enough to establish non negativity of this difference)